Solution Final Exam — Partial Differential Equations

9 April 2015, Aletta Jacobshal 02

Duration: 3 hours

Question 1 (15 points)

Consider the equation

$$u_x - 3x^2 u_y = 0, \tag{1}$$

where $u = u(x, y)$.

a. (7 pt) Find the general solution of Eq. (1).

b. (3 pt) Find the solution of Eq. (1) with the auxiliary condition $u(0, y) = -y^2$.

Consider now the equation

$$u_x - 3x^2 u_y = u. \tag{2}$$

c. (5 pt) Find the general solution of Eq. (2), using the substitution $u(x, y) = e^x w(x, y)$.

Solution

a. We solve the equation for the characteristic curves

$$\frac{dy}{dx} = -3x^2,$$

which directly gives

$$y = -x^3 + C,$$

where C is the constant of integration. Solving for C we get

$$C = y + x^3.$$

Since $y + x^3$ is constant along the characteristic curves we conclude that the solution of the problem has the general form

$$u(x, y) = f(y + x^3),$$

where f is an arbitrary function of one variable.

b. Applying the general solution we find

$$u(0, y) = f(y) = -y^2.$$

Therefore $f(s) = -s^2$, and the solution we are after is

$$u(x, y) = -(y + x^3)^2.$$

c. We have

$$u_x = e^x w + e^x w_x, \quad u_y = e^x w_y.$$

Then Eq. (2) gives

$$e^x(w + w_x - 3x^2 w_y - w) = 0,$$

which can be simplified, to

$$w_x - 3x^2 w_y = 0,$$

which is exactly Eq. (1), and for which we know that the general solution is

$$w = f(y + x^3).$$

Therefore, the general solution for Eq. (2) is

$$u = e^x f(y + x^3),$$

with f an arbitrary function.

Question 2 (15 points)

Consider the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0. \tag{3}$$

a. (3 pt) What is the type (elliptic / hyperbolic / parabolic) of Eq. (3)? Explain your answer.

b. (8 pt) Find a linear transformation $(x, y) \to (s, t)$ that reduces Eq. (3) to one of the standard forms $u_{ss} + u_{tt} = 0$, $u_{ss} - u_{tt} = 0$, or $u_{ss} = 0$.

c. (4 pt) Find the general solution of Eq. (3).

Solution

a. We have $a_{11} = 1$, $a_{22} = 4$, and $a_{12} = -2$. Therefore

$$a_{12}^2 = a_{11}a_{22},$$

and Eq. (3) is *parabolic*.

b. Since the equation is parabolic the standard form is $u_{ss} = 0$, or $\partial_s^2 u = 0$. Write the original equation as

$$\mathcal{L}u = 0,$$

where

$$\mathcal{L} = \partial_x^2 - 4\partial_x\partial_y + 4\partial_y^2.$$

Then

$$\mathcal{L} = (\partial_x - 2\partial_y)^2,$$

so we can set $\partial_s = \partial_x - 2\partial_y$ and $\partial_t = \partial_y$, that is,

$$\begin{pmatrix} \partial_s \\ \partial_t \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}.$$

The corresponding coordinate transformation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix},$$

or

$$x = s, \quad y = t - 2s,$$

which can be inverted to give

$$s = x, \quad t = y + 2x.$$

c. We have transformed Eq. (3) to $u_{ss} = 0$. The latter has the general solution

$$u(s, t) = f(t) + sg(t).$$

This means that the original equation has the solution

$$u(x, y) = f(y + 2x) + xg(y + 2x).$$

Question 3 (15 points)

Consider the equation for the damped string

$$u_{tt} - c^2 u_{xx} + r u_t = 0, \tag{4a}$$

where $x \in [0, L]$, $t \geq 0$, $c > 0$, $r > 0$, $L > 0$, and Dirichlet boundary conditions

$$u(0,t) = u(L,t) = 0. \tag{4b}$$

Define the energy of the string as

$$E(t) = \frac{1}{2}\int_0^L (u_t^2 + c^2 u_x^2)\, dx. \tag{5}$$

- **a. (8 pt)** Prove that the energy of the string decreases with time, that is, $dE/dt \leq 0$.
- **b. (7 pt)** Prove uniqueness of solutions u that satisfy Eq. (4a), Eq. (4b), and $u(x,0) = g(x)$, $u_t(x,0) = h(x)$ for $x \in [0, L]$.

Solution

a. We compute

$$\frac{dE}{dt} = \frac{1}{2}\int_0^L \frac{\partial}{\partial t}(u_t^2 + c^2 u_x^2)\, dx = \int_0^L (u_t u_{tt} + c^2 u_x u_{xt})\, dx.$$

Using Eq. (4a) we rewrite the last expression as

$$\frac{dE}{dt} = \int_0^L (u_t c^2 u_{xx} + c^2 u_x u_{xt} - r u_t^2)\, dx = c^2 \int_0^L (u_t u_{xx} + u_x u_{xt})\, dx - r\int_0^L u_t^2\, dx.$$

Since $(u_t u_x)_x = u_t u_{xx} + u_x u_{xt}$ we have

$$\frac{dE}{dt} = c^2 \int_0^L (u_t u_x)_x\, dx - r\int_0^L u_t^2\, dx = c^2[(u_t u_x)|_{x=L} - (u_t u_x)|_{x=0}] - r\int_0^L u_t^2\, dx,$$

so

$$\frac{dE}{dt} = c^2[u_t(L,t)u_x(L,t) - u_t(0,t)u_x(0,t)] - r\int_0^L u_t^2\, dx.$$

From the definition of partial derivatives we have for fixed $x = a$ that

$$u_t(a,t) = \frac{d}{dt}[u(a,t)].$$

In particular,

$$u_t(0,t) = \frac{d}{dt}[u(0,t)] = \frac{d}{dt}[0] = 0,$$

and similarly $u_t(L,t) = 0$. Therefore,

$$\frac{dE}{dt} = -r\int_0^L u_t^2\, dx.$$

Since $u_t^2 \geq 0$ we also have $\int_0^L u_t^2\, dx \geq 0$ and since $r > 0$ we finally get

$$\frac{dE}{dt} \leq 0.$$

b. Consider two solutions u_1, u_2 that satisfy Eqs. (4a) and (4b). Define $w = u_1 - u_2$. Then

$$w_{tt} - c^2 w_{xx} + r w_t = 0,$$

and

$$w(x,0) = u_1(x,0) - u_2(x,0) = g(x) - g(x) = 0,$$

$$w_t(x,0) = (u_1)_t(x,0) - (u_2)_t(x,0) = h(x) - h(x) = 0.$$

Then for the energy $E(t)$ corresponding to w we have $dE/dt \leq 0$ and

$$E(0) = \frac{1}{2}\int_0^L (w_t(x,0)^2 + c^2 w_x(x,0)^2)\,dx.$$

We have $w_x(x,0) = \frac{d}{dx}[w(x,0)] = 0$ and we also saw that $w_t(x,0) = 0$, so

$$E(0) = 0.$$

Since $dE/dt \leq 0$ we conclude that $E(t) \leq 0$ for all $t \geq 0$. Furthermore, by its definition $E(t) \geq 0$ so we conclude that $E(t) = 0$. This implies that $w_t(x,t) = w_x(x,t) = 0$ for all $t \geq 0$ and $x \in [0, L]$. Therefore w is constant and since at $t = 0$ it is $w(x,0) = 0$ we conclude that $w(x,t) = 0$. From here $u_1(x,t) = u_2(x,t)$.

Question 4 (20 points)

Consider the Laplace equation $\Delta u = 0$ in a half-disk H of radius a, that is,

$$H = \{(x, y) \, : \, x^2 + y^2 \le a^2, \, y \ge 0\},$$

with the boundary conditions $u(a, \theta) = \sin\theta$ for $0 \le \theta \le \pi$, and $u(r, 0) = u(r, \pi) = 0$ for $0 \le r \le a$.

a. **(5 pt)** Separate the Laplace equation in polar coordinates r, θ using the ansatz $u(r, \theta) = R(r)\Theta(\theta)$ and write two ordinary differential equations, one for R and one for Θ.

b. **(7 pt)** Solve the eigenvalue equation for Θ for the given boundary conditions (find eigenvalues and eigenfunctions). Consider known that the problem has no complex eigenvalues but check for positive, negative, or zero eigenvalues.

c. **(4 pt)** Solve the differential equation for R.

d. **(4 pt)** Write the general solution $u(r, \theta)$ for arbitrary boundary conditions $u(a, \theta) = h(\theta)$ and then give the solution for the specific boundary conditions in this problem.

Solution

a. Substituting $u(r, \theta) = R(r)\Theta(\theta)$ into the equation

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

we get

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Then

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0.$$

and separating the parts that depend on r from those that depend on θ we find

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda,$$

where λ is a constant. Then we have the two equations

$$r^2R'' + rR' - \lambda R = 0, \quad \Theta'' + \lambda\Theta = 0.$$

b. The given boundary conditions imply

$$\Theta(0) = \Theta(\pi) = 0.$$

For positive eigenvalues $\lambda = \beta^2$ we have the solutions

$$\Theta(\theta) = A\cos(\beta\theta) + B\sin(\beta\theta).$$

From here

$$\Theta(0) = A = 0, \quad \Theta(\pi) = A\cos(\beta\pi) + B\sin(\beta\pi) = 0.$$

Then

$$A = 0, \quad B\sin(\beta\pi) = 0,$$

and finally

$$\beta_n = n,\ n = 1, 2, 3, \ldots$$

For $\lambda = 0$ the solution is

$$\Theta(\theta) = A\theta + B$$

so

$$\Theta(0) = B = 0, \quad \Theta(\pi) = A\pi + B = 0$$

giving the trivial solution $A = B = 0$ which is rejected.
For $\lambda = -\gamma^2 < 0$ we have

$$\Theta(\theta) = Ae^{\gamma\theta} + Be^{-\gamma\theta}.$$

Then

$$\Theta(0) = A + B = 0, \quad \Theta(\pi) = Ae^{\gamma\pi} + Be^{-\gamma\pi} = 0,$$

so

$$B = -A, \quad Ae^{-\gamma\pi}(e^{2\gamma\pi} - 1) = 0.$$

The last equation implies that either $\gamma = 0$ (so $\lambda = 0$ but we assumed $\lambda < 0$) or $A = B = 0$ giving the trivial solution so we should also reject the case of negative eigenvalues.
Finally, the eigenvalues are

$$\lambda_n = \beta_n^2 = n^2,$$

and the eigenfunctions

$$\Theta_n(\theta) = \sin(n\theta).$$

c. The differential equation for R is $r^2R'' + rR' - n^2R = 0$. Try the solution $R = r^\alpha$ which gives $\alpha = \pm n$. Therefore

$$R_n(r) = A_n r^n + B_n r^{-n}.$$

Since we want the solutions to be well-defined at $r = 0$ we set $B_n = 0$.

d. The general solution is

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta).$$

For $r = a$ we have

$$u(a, \theta) = \sum_{n=1}^{\infty} A_n a^n \sin(n\theta) = h(\theta)$$

so

$$A_n = \frac{2}{\pi a^n} \int_0^\pi h(\theta) \sin(n\theta).$$

In the specific case here we have

$$h(\theta) = \sin\theta = \sum_{n=1}^{\infty} A_n a^n \sin(n\theta),$$

and comparing the two expressions we see that $A_1 = 1/a$, $A_n = 0$ in all other cases. Therefore the solution for the specific boundary conditions is

$$u(r, \theta) = \frac{r}{a} \sin\theta.$$

Question 5 (10 points)

Suppose that u is a harmonic function in the closed disk $D = \{r \leq 1\}$ and that $u = 2\cos(3\theta)+1$ for $r = 1$.

a. (5 pt) What are the maximum and minimum values of u in D?

b. (5 pt) Find the value of u at the origin.

Solution

a. Since u is harmonic it attains its maximum and minimum values at the boundary. At the boundary we have $u = 2\cos(3\theta) + 1$. Since $-1 \leq \cos(3\theta) \leq 1$ and the values ± 1 are attained for $\theta \in [0, 2\pi]$ we conclude that $-1 \leq u \leq 3$ at the boundary and u attains the maximum value 3 and the minimum value -1 at some point on the boundary. Therefore these are also the respective maximum and minimum values on D.

b. Poisson's formula is

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\, d\phi.$$

Applying Poisson's formula for $r = 0$, $a = 1$, $h(\phi) = 2\cos(3\phi) + 1$ gives

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} (2\cos(3\phi) + 1)\, d\phi = 1.$$

Question 6 (15 points)

Consider the function

$$f(x) = \pi - x, \quad \text{with} \quad x \in [0, \pi],$$

and its Fourier sine series.

a. **(2 pt)** Does the Fourier sine series converge in the L^2 sense? Explain your answer.

b. **(3 pt)** What is the pointwise limit of the Fourier sine series for $x \in \mathbb{R}$?

c. **(3 pt)** How does the Gibbs phenomenon manifest itself in the Fourier sine series? That is, at which point(s) in $[0, \pi]$ the Gibbs phenomenon appears and approximately how much is the "overshoot" there?

d. **(7 pt)** Compute the coefficients of the Fourier sine series for $f(x)$.

Solution

a. The function f is bounded in $[0, \pi]$, therefore

$$\|f\|^2 = \int_0^{\pi} f(x)^2 \, dx < +\infty.$$

This means that the Fourier sine series converges in the L^2 sense.

b. The pointwise limit of the Fourier sine series can be deduced from the odd-periodic extension $f_{\text{ext}}(x)$ of $f(x)$ from $[0, \pi]$ to $\mathbb{R}$. This is constructed by first considering the extension of f to an odd function defined in $[-\pi, \pi]$ and then the further periodic extension to $\mathbb{R}$. This extension $f_{\text{ext}}(x)$ is discontinuous at $x = 2k\pi$, $k \in \mathbb{Z}$ and $f_{\text{ext}}(2k\pi^+) = \pi$ while $f_{\text{ext}}(2k\pi^-) = -\pi$. Therefore the Fourier sine series converges pointwise at $x = 2k\pi$ to $\frac{1}{2}[f_{\text{ext}}(2k\pi^+) + f_{\text{ext}}(2k\pi^-)] = 0$. At all other $x \in \mathbb{R}$, $f_{\text{ext}}(x)$ is continuous so the Fourier series converges pointwise to $f_{\text{ext}}(x)$.

c. The odd-periodic extension f_{ext} of f is discontinuous at $x = 2k\pi$, $k \in \mathbb{Z}$ so the only point in $[0, \pi]$ where f_{ext} is discontinuous is $x = 0$. The jump of f_{ext} at $x = 0$ is $f_{\text{odd}}(0^+) - f_{\text{odd}}(0^-) = 2\pi$. Therefore, for $x \in [0, \pi]$ the Gibbs phenomenon appears at $x = 0$ and the overshoot is approximately $0.09 \cdot (2\pi) \simeq 0.56$.

d. We have

$$A_n = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \sin(nx) \, dx = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \left(-\frac{1}{n} \cos(nx) \right)' \, dx.$$

Integration by parts gives

$$A_n = -\frac{2}{n\pi} \Big[(\pi - x) \cos(nx) \Big]_0^{\pi} - \frac{2}{n\pi} \int_0^{\pi} \cos(nx) \, dx = -\frac{2}{n\pi} \left[(\pi - x) \cos(nx) + \frac{\sin(nx)}{n} \right]_0^{\pi}.$$

Then

$$A_n = \frac{2}{n}.$$